The Prime Number Theorem and Approximations For $\pi(x)$

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I. Background

Number Theory is one of the celebrated and historic areas of the vast study of Mathematics. Although no definitive date is known when humans first began studying Number Theory, it was clearly used by the Greeks over 2500 years ago. Among the first topics studied in Number Theory was the "natural numbers" or counting numbers such as 1, 2, 3… Among the natural numbers, a subset of numbers was discovered to contain a special property. "Prime numbers" are natural numbers that are only divisible by 1 and itself. For example, 5 is a prime number because the only natural numbers that 5 is divisible by are 1 and 5. A list of the smallest prime numbers is as follows: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31…

At first appearance of this list, there appears to be no simple formula for deriving the next prime number, as no apparent pattern exists. Another mystery involving the prime numbers was how many primes existed. The ancients studying prime numbers asked this very question and around 300 B.C. Euclid proved that there existed an infinite number of prime numbers. Euclid's argument went as follows. Suppose there are a finite number of prime numbers, $p_1, p_2, p_3, \ldots, p_n$. Now consider the number $x=p_1*p_2*\ldots*p_n+1$. Dividing x by any of these prime numbers will result in a remainder of 1, so none of these prime numbers divides x. This implies that x is prime, which is a contradiction that we had already listed all prime numbers. From this, we determine that our list was incomplete and in fact, there are an infinite number of prime numbers.

Although this proof by Euclid provided information that an infinite number of prime numbers existed, it gave no indication of how the prime numbers were distributed.

Let $\pi(x)$ be the number of prime numbers not greater than x, also known as the prime counting function. The behavior of $\pi(x)$ is extremely erratic and it took hundred of years, before any mathematician got a grasp on how $\pi(x)$ behaved. Adrien-Marie Legendre conjectured a series of approximations for $\pi(x)$ in 1798. First among his approximations was $\pi(x) \approx x/(\log(x))$ and soon later, he also suggested $\pi(x) \approx x/(\log(x)-1)$. These approximations essentially purposed that the distribution of primes followed a logarithmic function and as x becomes larger, primes become less common. Mathematicians had long conjured that primes were much less densely distributed when x becomes large, so Legendre's approximations affirmed this belief. In 1793, Carl Friedrich Gauss had apparently conjectured $\pi(x) \approx x/(\log(x))$ at the age of 14 independent of Legendre, although this was not known until years later. Gauss, however, proposed a brilliant new approximation for $\pi(x)$. He purposed $\pi(x) \approx \mathrm{li}(x)$.

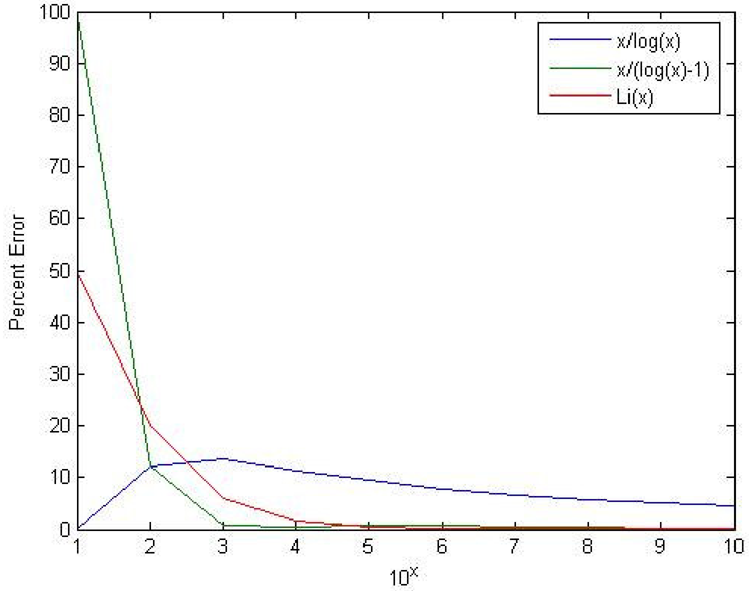
$$\mathrm{li}(x) := \int_2^x \frac{1}{\log(t)}\,dt$$

Li(x) is also commonly known as the logarithmic integral. With this estimation seemingly more accurate than Legendre's approximations, he attempted to refine his estimate and devised that $\pi(x) \approx x/(\log(x)+1.0836)$. This estimation was more accurate for numbers $x<10^9$, but becomes more inaccurate for values of $x>10^9$. Later, Riemann would devise a function that would ultimately prove to even more accurate and is named after him for this development. This will be explained more in depth later. Next, we will examine how effective these approximations are.

II. Accuracy of Prime Counting Approximations

Today, values of π(x) have been accurately calculated for very large values of x by using modern supercomputers. Using these known values of π(x), below is an investigation into the accuracy of the approximation methods described above. The values for the approximation methods were constructed using a program written in MATLAB and details of this program are included in the Appendix.

x	pi(x)	x/log(x)	Error(%)	x/(log(x)-1)	Error(%)	Li(x)	Error(%)
10^1	4	4	0.0	8	100.0	6	50.0
10^2	25	22	12.0	28	12.0	30	20.0
10^3	168	145	13.7	169	0.6	178	6.0
10^4	1225	1086	11.3	1218	0.6	1246	1.7
10^5	9592	8686	9.4	9512	0.8	9630	0.4
10^6	78498	72382	7.8	78030	0.6	78628	0.2
10^7	664579	620421	6.6	661459	0.5	664918	0.1
10^8	5761455	5428681	5.8	5740304	0.4	5762209	0.0
10^9	50847534	48254942	5.1	50701542	0.3	50849235	0.0
10^10	455052511	434294482	4.6	454011971	0.2	455055615	0.0



As seen above, Legendre's second approximation for π(x) ≈ x/(log(x)-1) is quite an improvement over his original approximation π(x) ≈ x/log(x). Additionally, it is seen above that Gauss' approximation π(x) ≈ Li(x) has an error less than 0.1% when $x=10^8$. In fact, all of these approximations become better as x→ ∞. This is the basis for the Prime Number Theorem.

III. The Prime Number Theorem

Although numerous approximations were detailed above, the two most famous approximations for π(x) are Li(x) and x/log(x). Although the Prime Number Theorem is often stated in different terms, the most general states that:

$$\lim_{x \to \infty} \frac{\pi(x)}{\left(\dfrac{x}{\log(x)}\right)} = \lim_{x \to \infty} \frac{\pi(x)}{\mathrm{li}(x)} = 1$$

This can also be stated as follows:

x/log(x) ¬ li(x) ¬ π(x) as x→ ∞

(where a(x) ¬ b(x) iff b(x)/a(x) = 1 as x→ ∞)

Before continuing, here is a useful theorem that is generally proven in undergraduate analysis courses.

<u>Theorem</u>: If a(x) ¬ b(x) and b(x) ¬ c(x), then a(x) ¬ c(x) (i.e. if a(x) converges to b(x) and b(x) converges to c(x) then a(x) converges to c(x)).

This is helpful as follows: We can first prove that x/log(x) ¬ li(x) as x→ ∞. A proof of this will be given in the Appendix. Next, we can prove that li(x) ¬ π(x) as x→ ∞. This will be outlined shortly. From these two statements and the theorem provided above, we will have proven the Prime Number Theorem.

Major success involving the Prime Number Theorem was made by a Russian mathematician/statistician named Pafnuty Lvovich Chebyshev. Chebyshev is most known for his work in Probability Theory, but he was also very interested in the Prime Number Theorem. Specifically, he was extremely successful on establishing bounds for the functions. His ultimate goal was to prove that there exists $C_1 > C_2 > 0$ such that for all $x \geq 2$, we have:

$$C_2\, x/\log(x) \leq \pi(x) \leq C_1\, x/\log(x)$$

To prove this, Chebyshev began by defining the following functions:

$$\theta(x) = \left[\sum_{(p \leq x)} \log(p)\right] \quad \text{where p is prime}$$

$$\psi(x) = \sum_{n=1}^{\infty} \theta\left[x^{\left(n^{-1}\right)}\right]$$

First, he began by proving that there exists A_1, A_2, A_3 such that:

$A_2\, x \leq \psi(x) \leq A_1\, x$,which was shown to prove that:

$A_3\, x \leq \theta(x) \leq A_1\, x$

Finally, Chebyshev provided a lemma that:

(i) $\theta(x) \leq \log(x) * \pi(x)$

(ii) $\theta(x) \leq \pi(x) - x*\log(x)$

These bounds along with $A_3\, x \leq \theta(x) \leq A_1\, x$, provided the ultimate goal that there exists $C_1 > C_2 > 0$ such that:

$$C_2\, x/\log(x) \leq \pi(x) \leq C_1\, x/\log(x)$$

In fact, using the above theorem and lemma, Chebyshev produced the following bounds:

$$.921^* \, x/\log(x) \le \pi(x) \le 1.105 \, x/\log(x)$$

In 1881, J.J. Sylvester improved these bounds with a much more complicated method.

These bounds were groundbreaking, as those showed that $\pi(x)$ never varied far from $x/\log(x)$, but at the same time, they failed to show that as $x \to \infty$, $x/\log(x) \neg \pi(x)$. Clearly, however, it seemed to imply that the $x/\log(x) \neg \pi(x)$ was true, as was later proven.

An additional result was found in Chebyshev's papers. Chebyshev's proof established that $\psi(x)/x$ and $\theta(x)/x$ has the same limit.

$$\lim_{x \to \infty} \frac{1}{x} \cdot (\psi(x) - \theta(x)) = 0$$

Additionally, Chebyshev showed the following:

$$\lim_{x \to \infty} \frac{\psi(x)}{x} = 1$$

These results helped to establish $a\,x \le \theta(x) \le b\,x$. He used this statement in his quest for finding bounds on $C_2 \, x/\log(x) \le \pi(x) \le C_1 \, x/\log(x)$. Although his work on bounds was finished, he was unknowingly one statement away from proving the Prime Number Theorem. Soon after Chebyshev's work on the bounds, the last essential element was shown to finish the proof on the Prime Number Theorem.

Given: there exists a, b such that: $a\,x \le \theta(x) \le b\,x$, the following are also true:

(i) $$a\cdot\left(\mathrm{li}(x) + \frac{2}{\log(2)}\right) \le \pi(x) \le b\cdot\left(\mathrm{li}(x) + \frac{2}{\log(2)}\right)$$

(ii) Let $\varepsilon>0$, there exists $x>x1$ so that:

$$(a-\varepsilon)\cdot\mathrm{li}(x) \le \pi(x) \le (b-\varepsilon)\cdot\mathrm{li}(x)$$

Chebyshev had already shown that there exists a, b such that: $a\,x \le \theta(x) \le b\,x$, so it immediately followed that li(x) ¬ π(x) as x→ ∞, after the above was proved. Charles de La Vallée Poussin finished this proof of the Prime Number Theorem in 1899.

Although it isn't apparent from the steps shown above, some of the results above relied heavily on Complex Analysis and specifically, the zeta function defined below:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{s}} \text{ with } \mathrm{Re}(s) > 1$$

During the twentieth century, search began for a proof of the Prime Number Theorem using Number Theory rather Complex Analysis. Many sought such a proof and failed and even more believed that such a proof even existed. In 1949, Paul Erdõs and Atle Selberg completed a proof using only Number Theory. It was hailed for its brilliance and closed another chapter on the Prime Number Theorem.

IV. Beyond the Prime Number Theorem

Bernhard Riemann, most famous for suggesting the Riemann Hypothesis, also suggested a prime counting function. He conjectured that $\pi(x)=R(x)$, where

$$R(x) = \sum_{k=1}^{\log(k)} \frac{\mu(k)}{k} \cdot \mathrm{Li}\left(\sqrt[k]{x}\right)$$

And the Möbius function, $\mu(x)$, is defined as follows:

$\mu(1) = 1$
$\mu(x) = (-1)^n$ if $x = p_1 p_2 \ldots p_n$, a product of n distinct primes
$\mu(x) = 0$ if p^2 divides x for some prime p

Although $\pi(x) \neq R(x)$, it is a very good approximation for smaller values of x. J.E. Littlewood proved this conjecture false in 1914. Riemann also conjectured in 1859 the Riemann Hypothesis, which stated that are all non-trivial zeros for the Riemann zeta function lie on $\mathrm{Re}(s) = \frac{1}{2}$. This hypothesis leads to a better approximation to $\pi(x)$ by $\mathrm{Li}(x)$ than the known ones. In addition to its application in this context, many facts are already been proved to be true assuming the Riemann Hypothesis is also true. If the Riemann Hypothesis turns out to be false, the distribution of prime numbers does not follow the way we currently believe it does. Most mathematicians, however, believe in the truth of the Riemann Hypothesis, although it remains unproven after 150 years. When and if a proof or disproof of the Riemann Hypothesis is given, we will again learn more about the unpredictable behavior of the prime numbers.

V. Appendix

Part (i): Proof: li(x) ¬ x/log(x) as x→ ∞

The goal is to prove the above statement, but first a couple of helpful lemmas will be proven.

Definition: $I_n(x) = \displaystyle\int_e^x \frac{1}{(\log(t))^n}\,dt$ for n>0

This integral will begin at “e” rather than “2” to eliminate additional constant terms that will be eventually unimportant.

Recall: $\mathrm{li}(x) = \displaystyle\int_2^x \frac{1}{\log(t)}\,dt = \frac{x}{\log(x)} - \alpha + \int_2^x \frac{1}{(\log(t))^2}\,dt$

Carrying out integration by parts completes the right side of this equality.

$$I_n(x) = \int_e^x \frac{1}{(\log(t))^n}\,dt = \frac{x}{(\log(x))^n} - e + \int_e^x \frac{n}{(\log(t))^{n+1}}\,dt = \frac{x}{(\log(x))^n} - e + n\cdot I_{n+1}(x)$$

A similar integration by parts is also taken for the $I_n(x)$. We know will to prove a lemma that will help complete the proof.

Lemma: $I_n(x) \neg \dfrac{x}{(\log(x))^n}$ as $x \to \infty$

Proof:

The above statement is equivalent to proving:

$$\lim_{x \to \infty} I_{n+1}(x)\cdot\frac{(\log(x))^n}{x} = 0$$

First, divide up the integration into two parts. Also, we note that log(t) ≥ 1 for e ≤ t ≤ √x and log(t) ≥ ½log(x) for √x ≤ t ≤ x

$$I_{n+1}(x) = \int_{e}^{x^{\frac{1}{2}}} \frac{1}{(\log(t))^{n+1}}\,dt + \int_{x^{\frac{1}{2}}}^{x} \frac{1}{(\log(t))^{n+1}}\,dt < x^{\frac{1}{2}} + \left(\frac{2}{\log(x)}\right)^{n+1}$$

This overestimation allows the following statement:

$$I_{n+1}(x)\cdot\frac{x}{(\log(x))^{n}} < \frac{(\log(x))^{n}}{x^{\frac{1}{2}}} + \frac{2^{n+1}}{\log(x)}$$

<u>Recall:</u> $\lim_{x \to \infty} \frac{(\log(x))^{n}}{x} = 0$

So we have $\lim_{x \to \infty} I_{n+1}(x)\cdot\frac{(\log(x))^{n}}{x} = 0$ as required.

Using this lemma, we can proceed on with the main proof.

Let li(x) = $\frac{x}{\log(x)}$ + r(x), where r(x) is a remainder

Since $I_1(x)$ and li(x) only differ by a constant, li(x) ¬ $\frac{x}{\log(x)}$ iff $I_1(x)$ ¬ $\frac{x}{\log(x)}$

$I_1(x) = \frac{x}{\log(x)} + I_2(x) - e$

By the lemma, $I_2(x) - e$ ¬ $\frac{x}{(\log(x))^{2}}$

This term becomes zero as x→ ∞ and we have:

$I_1(x)$ ¬ $\frac{x}{\log(x)}$ and

li(x) ¬ $\frac{x}{\log(x)}$ as x→ ∞ as required.

Part (ii): MATLAB code for Section II

```
function estimations()
a=[4,25,168,1225,9592,78498,664579,5761455,50847534,455052511];
fprintf('x\tpi(x)\tx/log(x)\tError(%%)\tx/(log(x)-1)\tError(%%)\tLi(x)\tError(%%)\t\n');
for i=1:10
b(i)=10^i/log(10^i);
be(i)=100*abs((round((b(i))-a(i)))/a(i));
c(i)=10^i/(log(10^i)-1);
ce(i)=100*abs((round((c(i))-a(i)))/a(i));
d(i)=mfun('Li',10^i);
de(i)=100*abs((round(d(i))-a(i))/a(i));
fprintf('10^%i\t%10.0f\t%10.0f\t%2.1f\t%10.0f\t%02.1f\t%10.0f\t%2.1f\n',i,a(i),b(i),be(i)
,c(i),ce(i),d(i),de(i));
end;
plot(1:10,be,1:10,ce,1:10,de);
legend('x/log(x)','x/(log(x)-1)','Li(x)');
xlabel('10^x');
ylabel('Percent Error');
```

Part (iii): Author’s Note

The above proof (Appendix, Part (i)) and proof construction (Section III) were reproduced and some changes were made to the proofs for clarity. If either one is perplexing, please feel free to contact me for further clarity and/or information regarding the original sources. The MATLAB code, however, is completely original. Construction of the mathematical expressions was completed in MathCad. A list of sources is provided next.

Part (iv): Works Cited

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